$S \subseteq \mathbb{R}$, $a \in \mathbb{R}$ is an accumulation point of S if every neighborhood of a contains infinitely many points of S.

① i.e. $\forall \varepsilon > 0$, $\exists \{s_n\} \subseteq S$, s.t. $s_n \in (a-\varepsilon, a+\varepsilon)$, $\forall n$.

Over $\mathbb{R}$, it suffices to assert that every neighborhood of a contains at least one point of S other than a.

② i.e. $\forall \varepsilon > 0$, $\exists s \in S$, $s \neq a$, $s \in (a-\varepsilon, a+\varepsilon)$.

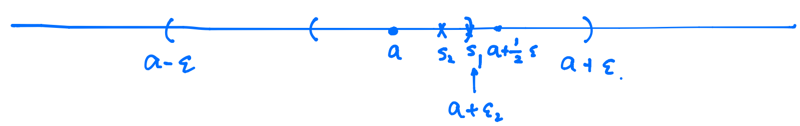
Why? Prove ② $\Rightarrow$ ①:

$\forall \varepsilon > 0$, $\exists s_1 \in S$, $s_1 \neq a$, $s_1 \in (a-\varepsilon, a+\varepsilon)$.

Pick $\varepsilon_2 = \min\left(\frac{1}{2}\varepsilon, |s_1 - a|\right)$

By ②, $\exists s_2 \in S$, $s_2 \neq a$, $s_2 \in (a-\varepsilon_2, a+\varepsilon_2) \subseteq (a-\varepsilon, a+\varepsilon)$

$s_2 \neq s_1$ because of the choice of ε_2.



Repeating the process: assume $s_1, s_2, \cdots, s_n$ is chosen.

Pick $\varepsilon_{n+1} = \min\left(\frac{1}{2^{n+1}}\varepsilon, |s_n - a|\right)$

By ②, $\exists s_{n+1} \in S$, $s_{n+1} \neq a$, $s_{n+1} \in (a-\varepsilon_{n+1}, a+\varepsilon_{n+1})$
$\subseteq (a-\varepsilon, a+\varepsilon)$

This way I have proved that

$\forall \varepsilon > 0$, $\exists \{s_n\} \subseteq S$. $s_n \in (a-\varepsilon, a+\varepsilon)$ $\forall n$.

i.e. ② $\Rightarrow$ ①. You can use ② as the def. of accumulation points.

<u>Example</u>: $\{(-1)^n\}$ has no accumulation points.

Recall: a is an accumulation point of S. if

$\forall \varepsilon > 0,\ \exists s \in S,\ s \neq a,\ \ s \in (a-\varepsilon, a+\varepsilon)$.

$\{(-1)^n\} = \{-1, 1\}$.

For $a=1$, $\exists \varepsilon > 0$, (e.g. $\varepsilon = \frac{1}{2}$), s.t. $\forall s \in S \cap (1-\varepsilon, 1+\varepsilon)$, $s = a$.

18. Find S such that S has two accumulation points:

① $\{1+\frac{1}{n}\} \cup \{\frac{1}{n}\}$

② $\{(-1)^n + \frac{1}{n}\} = \{-1 + \frac{1}{2k+1}\} \cup \{1 + \frac{1}{2k}\}$.

13. $x > 0$. $a_n = \frac{[x]+[2x]+\cdots+[nx]}{n^2}$, $[x]$ = the longest integer $\leq x$.

Prove that $\lim_{n\to\infty} a_n = \frac{x}{2}$.

Recall: $1+2+3+\cdots+n = \frac{n(n+1)}{2}$

Example: $1+2+3+4+5 = 6+6+\frac{1}{2}\cdot 6 = (5+1)(1+1+\frac{1}{2}) = (5+1)\cdot\frac{5}{2}$.

$\frac{5}{2}$ pairs of 6 added up together.

In general, $\frac{n}{2}$ pairs of $n+1$ added up together $\Rightarrow \frac{1}{2}n\cdot(n+1)$.

$1+2+3+4+5+6$ 3 pairs of 7 $\Rightarrow \frac{6}{2}\cdot 7$

Recall: $y-1 \leq [y] \leq y$ e.g. $x = 2.5$. $1.5 \leq [2.5] = 2 \leq 2.5$

Put $y = x, 2x, 3x, \cdots, nx$. into this inequality, then sum them up.

$$\underbrace{x+2x+3x+\cdots+nx}_{=\frac{n(n+1)}{2}x} - n \leq [x]+[2x]+[3x]+\cdots+[nx] \leq \underbrace{x+2x+3x+\cdots+nx}_{=\frac{n(n+1)}{2}x}$$

$$\frac{\frac{n(n+1)}{2}x - n}{n^2} \qquad \frac{[x]+[2x]+[3x]+\cdots+[nx]}{n^2} \qquad \frac{\frac{n(n+1)}{2}x}{n^2}$$

Squeeze Lemma yields the limit.

The proof is way trickier if you want to argue by definition:

$$\left| \frac{[x]+[2x]+\cdots+[nx]}{n^2} - \frac{x}{2} \right|$$

$\frac{x}{2} = \frac{n^2 x}{2n^2} = \frac{(n^2-n)}{2n^2}x + \frac{x}{2n}$

$$\leqslant \left| \frac{[x]+[2x]+\cdots+[nx]}{n^2} - \frac{n(n-1)}{2n^2}x \right| + \left| \frac{x}{2n} \right|$$

$\frac{n(n-1)}{2} = 1+2+\cdots+n$

$$= \left| \frac{[x]+[2x]+\cdots+[nx]}{n^2} - \frac{x+2x+\cdots+nx}{n^2} \right| + \frac{x}{2n}$$

$x>0,\ \left|\frac{x}{2n}\right| = \frac{x}{2n}$

$$= \left| \frac{[x]-x}{n^2} + \frac{[2x]-2x}{n^2} + \cdots + \frac{[nx]-nx}{n^2} \right| + \frac{x}{2n}$$

$$\leqslant \left|\frac{[x]-x}{n^2}\right| + \left|\frac{[2x]-2x}{n^2}\right| + \cdots + \left|\frac{[nx]-nx}{n^2}\right| + \frac{x}{2n}.$$

$|[y]-y| \leq 1$

$$\leqslant \frac{1}{n^2} + \frac{1}{n^2} + \cdots + \frac{1}{n^2} + \frac{x}{2n}$$

n copies of $\frac{1}{n^2}$ summed up

$$= \frac{1}{n} + \frac{x}{2n} = \frac{2+x}{2n}$$

So $\forall \varepsilon>0$, pick N such that $\frac{2+x}{2N} < \varepsilon \Rightarrow N = \left[\frac{2+x}{2\varepsilon}\right]+1$

then $\forall n>N$, $\left| \frac{[x]+[2x]+\cdots+[nx]}{n^2} - \frac{x}{2} \right| < \varepsilon.$

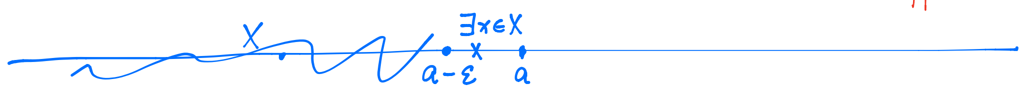
For workshop:

① Recall how sup was defined

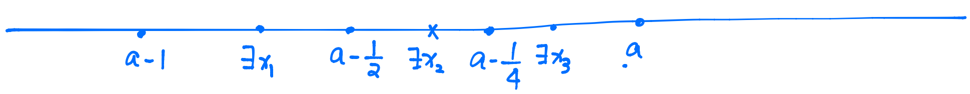
$X \subseteq \mathbb{R}$. $a = \sup X$ if

(i) $\forall x \in X$, $x \leq a$ (a is an upper bound)

(ii) $\forall \varepsilon > 0$, $\exists x \in X$, s.t. $x > a - \varepsilon$ ($a-\varepsilon$ is not an upper bound, $\forall \varepsilon > 0$)



② By choosing ε appropriately to find a sequence that approaches sup.



Apply (ii) to $\varepsilon = 1 \Rightarrow \exists x_1 \in X$, s.t. $x_1 > a - 1$.

$\varepsilon = \frac{1}{2} \Rightarrow \exists x_2 \in X$, s.t. $x_2 > a - \frac{1}{2}$

$\vdots \qquad \vdots \qquad \vdots$

$\varepsilon = \frac{1}{n} \Rightarrow \exists x_n \in X$, s.t. $x_n > a - \frac{1}{n}$.

$\vdots \qquad \vdots \qquad \vdots$

Result: Sequence $(x_n)_{n=1}^{\infty}$ satisfying $x_n > a - \frac{1}{n}$.

All $x_n \in X \Rightarrow x_n \leq a$.

Use the above to argue that $\lim_{n\to\infty} x_n = a$.